A Proofs of Main Results

A.1 Proof of Lemma 2.1

The necessity of (5) has been proved in the main text. On the other hand, fix a $d \in \mathcal{D}$, suppose (5) holds for some $z \in \mathcal{Z}$. Fix a particular $z^* \in \mathcal{Z}^*(d, Q)$ that satisfies (4). If $z = z^*$ then of course Assumption 2.2 holds for z. Suppose $z \neq z^*$. Then,

$$
\begin{aligned}
0 &\leq P\{D = d \mid Z = z\} - P\{D = d \mid Z = z^*\} \\
&= Q\{D_z = d\} - Q\{D_{z^*} = d\} \\
&= Q\{D_{z^*} \neq d, D_z = d\} - Q\{D_z \neq d, D_{z^*} = d\} \\
&= \ - Q\{D_z \neq d, D_{z^*} = d\} \ ,
\end{aligned}
$$

where the first inequality is using that z satisfies (5), the second line is using Assumption 2.1, and the last line is using that z^* satisfies (4). Thus $Q\{D_z \neq d, D_{z^*} = d\} = 0$. Assumption 2.2 implies

$$
\begin{aligned}
&Q\{D_z \neq d, D_{z'} = d \text{ for some } z' \in \mathcal{Z}\} \\
&= Q\{D_z \neq d, D_{z^*} = d\} + Q\{D_z \neq d, D_{z^*} \neq d, D_{z'} = d \text{ for some } z' \in \mathcal{Z} \setminus \{z^*\}\} \\
&= 0 \ ,
\end{aligned}
$$

and z satisfies Assumption 2.2 as well. ■

A.2 Proof of Theorem 3.1

The desired result follows immediately from Theorems 3.2 and 3.3. ■

A.3 Proof of Theorem 3.2

This section is organized as follows. In Section A.3.1, we introduce additional notation that is helpful in formally proving our result, including defining subgroups of individuals, called treatment response types, who are defined by what treatment they would take at each value of the instrument. Because all variables are discrete, we will directly work with the probability mass function. We derive a lemma that characterizes a sufficient condition for a given distribution of potential outcomes and treatments Q to rationalize the distribution of the data P. The lemma states that whether a Q that satisfies Assumption 2.1 rationalizes P depends on the probability of each treatment response type and on the probability of that type's potential outcomes corresponding to treatments they would choose for some value of the instrument (so that they "comply with" this treatment at least for some values of the instrument), but does not depend on the probability of that type's potential outcomes corresponding to treatments they would not choose for any value of the instrument (so that they are "never-takers" of this treatment). If Q satisfies Assumptions 2.1 and 2.2, then the set of treatments for which a treatment response type is a "never-taker" are precisely the set of treatments that they would not take even when maximally encouraged to do so. Therefore, the implication of the lemma is that if Q satisfies Assumptions 2.1 and 2.2 and rationalizes P, then any other Q^* satisfying Assumption 2.1 and 2.2 will also rationalize P if Q and Q^* differ *only* in the probability of potential outcomes corresponding to treatments that a given response type would not take even when maximally encouraged to do so.

In Section A.3.2, we use the notation and lemma introduced in Section A.3.1 to prove the theorem. Let $\mathbf{Q}$ satisfy the assumptions of the theorem. We first show that $\Theta_0(P, \mathbf{Q})$ is a subset of the bounds in (6). We then show that the bounds in (6) are a subset of $\Theta_0(P, \mathbf{Q})$ using the following proof strategy. By assumption, $\Theta_0(P, \mathbf{Q})$ is non-empty, so that there exists a distribution $Q \in \mathbf{Q}$ that rationalizes P. For each value θ_0 in the bounds of (6), we construct an alternative distribution $Q^* \in \mathbf{Q}$ such that $\theta(Q^*) = \theta_0$ with Q and Q^* differing only in the probability of outcomes corresponding to treatments that a given response type would not take even when maximally encouraged to do so. That the constructed Q^* lies in $\mathbf{Q}$ follows from the assumption that $\mathbf{Q}$ satisfies Assumption 3.1 and that Q and Q^* have the same distribution of potential treatment choices with $Q \in \mathbf{Q}$. That the constructed Q^* rationalizes P follows from $Q \in \mathbf{Q}$ and the previously described lemma. That we are able to construct such a Q^* with $\theta(Q^*) = \theta_0$ for every θ_0 in the bounds of (6) establishes that the bounds (6) are a subset of $\Theta_0(P, \mathbf{Q})$, completing the proof.

A.3.1 Auxillary Results

To present the proof of Theorem 3.2, we first introduce some further notation. Because all variables are discrete, we will directly work with the probability mass function. Recall from the discussion in Section 2 that if Q satisfies Assumption 2.1, then $P = QT^{-1}$ if and only if

$$p_{yd|z} = Q\{Y_d = y, D_z = d\} \ .$$

Following Heckman and Pinto (2018), we define a treatment response type as a vector $r^t \in \mathcal{D}^{|\mathcal{Z}|}$,

$$r^t = (d_z : z \in \mathcal{Z}) \in \mathcal{D}^{|\mathcal{Z}|} \ .$$

Treatment response types are also called principal strata (Frangakis and Rubin, 2002). We analogously define an outcome response type as a vector $r^o \in \mathcal{Y}^{|\mathcal{D}|}$,

$$r^o = (y_d : d \in \mathcal{D}) \in \mathcal{Y}^{|\mathcal{D}|} \ .$$

Because all variables are discrete, we define the probability mass function as

$$q\big(r^o, r^t\big) = Q\{(Y_d : d \in \mathcal{D}) = r^o, (D_z : z \in \mathcal{Z}) = r^t\} \ .$$

For the rest of the proof, without loss of generality, we suppose $\mathcal{D} = \{0, 1, \ldots, |\mathcal{D}| - 1\}$ and $\mathcal{Z} = \{0, 1, \ldots, |\mathcal{Z}| - 1\}$. Let r^o_j denote the $(j+1)$th entry of r^o and r^t_j denote the $(j+1)$th entry of r^t. In other words, r^o_j denotes the value of the potential outcome for the outcome response type r^o when the treatment equals j, and r^t_j denotes the value of the potential treatment for the treatment response type r^t when the instrument equals j. In this notation, if Q satisfies Assumption 2.1, then it follows from (2) that $P = QT^{-1}$ if and only if

$$p_{yd|z} = \sum_{(r^o, r^t) : r^o_d = y, r^t_z = d} q(r^o, r^t) \qquad \forall\, y \in \mathcal{Y},\ d \in \mathcal{D},\ z \in \mathcal{Z}\ . \tag{S.1}$$

Below we derive a lemma that simplifies determining whether $q(r^o, r^t)$ satisfies (S.1) and will be used subsequently to derive our characterization of the identified set. To this end, we require some further notation. Let

$$\mathcal{N}(r^t) = \{d \in \mathcal{D}\ :\ r^t_z \neq d \text{ for all } z \in \mathcal{Z}\}\ ,$$
$$\mathcal{N}(r^t)^c = \{d \in \mathcal{D}\ :\ r^t_z = d \text{ for some } z \in \mathcal{Z}\}\ ,$$

For a given treatment response type r^t, $\mathcal{N}(r^t)$ is the set of treatments for which that treatment response type is a "never-taker," and $\mathcal{N}(r^t)^c$ is the set of treatments for which that treatment response type will "comply with" the treatment for some value of z. Using this notation, partition outcome and treatment response types (r^o, r^t) as $(r^o_n(r^t), r^o_c(r^t), r^t)$ where

$$r^o_n(r^t) = (r^o_d : d \in \mathcal{N}(r^t))\ ,$$
$$r^o_c(r^t) = (r^o_d : d \in \mathcal{N}(r^t)^c)\ .$$

For a given treatment response type r^t, $r^o_n(r^t)$ are those outcomes that are never observed for that response type, and $r^o_c(r^t)$ are the remaining outcomes that are observed given some value of Z. Here, the subscripts n and c stand for "never-taker" and "complier."

Remark A.1. Here we illustrate how our notation specializes under Assumption 2.2. Note Assumption 2.2 can be expressed as restricting $q(r^o, r^t) = 0$ unless the treatment response type r^t satisfies the condition therein; in other words, it restricts the support of the treatment response type. In particular, if for some $d \in \mathcal{D}$, $r^t_{z^*(d)} \neq d$ while $r^t_{z'} = d$ for some $z' \neq z^*(d)$, then $q(r^o, r^t) = 0$ for all r^o. For any r^t in the support,

$$\mathcal{N}(r^t) = \{d \in \mathcal{D} \ : \ r^t_{z^*(d)} \neq d\} \ ,$$
$$\mathcal{N}(r^t)^c = \{d \in \mathcal{D} \ : \ r^t_{z^*(d)} = d\} \ ,$$

and

$$r^o_n(r^t) = (r^o_d : d \in \mathcal{D}, \ r^t_{z^*(d)} \neq d) \ ,$$
$$r^o_c(r^t) = (r^o_d : d \in \mathcal{D}, \ r^t_{z^*(d)} = d) \ .$$

Indeed, $z^*(d)$ is the instrument that maximally encourages to treatment d, so if $r^t_{z^*(d)} \neq d$, then $r^t_z \neq d$ for all $z \in \mathcal{Z}$. As a result, someone with that treatment response type r^t never takes d, and hence $d \in \mathcal{N}(r^t)$. Otherwise, $d \in \mathcal{N}(r^t)^c$, or this person is a "complier" for treatment d at least when $z = z^*(d)$. The outcome response type r^o is then partitioned into $r^o_n(r^t)$ and $r^o_c(r^t)$ according to whether $d \in \mathcal{N}(r^t)$ or not. ■

For notational convenience, we further define the probability mass $q(r^o_c(r^t), r^t)$ as $q(r^o_n(r^t), r^o_c(r^t), r^t)$ summed over $r^o_n(r^t)$:

$$q(r^o_c(r^t), r^t) = \begin{cases} q(r^o, r^t) & \text{if } \mathcal{N}(r^t) = \emptyset \text{ so that } r^o_c(r^t) = r^o \\ \sum_{r^o_n(r^t) \in \mathcal{Y}^{|\mathcal{N}(r^t)|}} q(r^o_n(r^t), r^o_c(r^t), r^t) & \text{if } \mathcal{N}(r^t) \neq \emptyset \text{ so that } r^o_c(r^t) \neq r^o \ . \end{cases}$$

In defining the probability mass $q(r^o_c(r^t), r^t)$, we sum over all possible values of $r^o_n(r^t)$, because these are the outcomes of treatments that are never taken by the treatment response type r^t, and hence will not be relevant for the observed data. Using this notation, we have the following lemma that asserts whether $q(r^o, r^t)$ satisfies (S.1) depends only on $q(r_c^o(r^t), r^t)$. This lemma implies that whether a distribution of potential outcomes and treatments Q that satisfies Assumption 2.1 rationalizes the distribution of the data P depends only on the probability of each treatment response type and the probability of that type's potential outcomes that would be observed for some value of the instrument.

Lemma A.1. *Suppose q satisfies* (S.1)*. Then, q^* satisfies* (S.1) *if, for each $r^t \in \mathcal{D}^{|\mathcal{Z}|}$,*

$$q^*(r_c^o(r^t), r^t) = q(r_c^o(r^t), r^t) \quad \forall \; r_c^o(r^t) \; . \tag{S.2}$$

Proof. We can rewrite (S.1) as

$$\begin{aligned} p_{yd|z} &= \sum_{r^t : r_z^t = d} \; \sum_{r^o : r_d^o = y} q(r^o, r^t) \\ &= \sum_{r^t : r_z^t = d} \; \sum_{r_c^o(r^t) : r_d^o = y} \left(\sum_{r_n^o(r^t)} q(r_n^o(r^t), r_c^o(r_t), r^t) \right) \\ &= \sum_{r^t : r_z^t = d} \; \sum_{r_c^o(r^t) : r_d^o = y} q(r_c^o(r_t), r^t) \; , \end{aligned}$$

where the second equality uses that $r_c^o(r^t)$ is nonempty because $r_z^t = d$ and that r_d^o is an element of $r_c^o(r^t)$ for r^t such that $r_z^t = d$. The result now follows. ■

A.3.2 Proof of the Theorem

$\Theta_0(P, \mathbf{Q}) \subseteq (6)$

We first show that (6) provides valid bounds on $\theta(Q)$ under the stated assumptions, that is, $\Theta_0(P, \mathbf{Q})$ is a subset of the bounds of (6). Suppose $Q \in \mathbf{Q}_0(P, \mathbf{Q})$. For each $d \in \mathcal{D}$,

$$\begin{aligned} \mathbb{E}_Q[Y_d] &= \mathbb{E}_Q[Y_d \mathbb{1}\{D_{z^*(d)} = d\}] + \mathbb{E}_Q[Y_d \mathbb{1}\{D_{z^*(d)} \neq d\}] \\ &= \beta_{d|z^*(d)} + \mathbb{E}_Q[Y_d \mathbb{1}\{D_{z^*(d)} \neq d\}] \; , \end{aligned}$$

where the second equality is using Assumption 2.1. We have

$$\mathbb{E}[Y_d \mathbb{1}\{D_{z^*(d)} \neq d\}] \in [y^L Q\{D_{z^*(d)} \neq d\},\ y^U Q\{D_{z^*(d)} \neq d\}] ,$$

while Assumption 2.1 implies that

$$Q\{D_{z^*(d)} \neq d\} = 1 - \sum_{y \in \mathcal{Y}} p_{yd|z^*(d)} .$$

We thus have that

$$\mathbb{E}[Y_d] \in [\beta_{d|z^*(d)} + y^L(1 - \sum_{y \in \mathcal{Y}} p_{yd|z^*(d)}),\ \beta_{d|z^*(d)} + y^U(1 - \sum_{y \in \mathcal{Y}} p_{yd|z^*(d)})] ,$$

for each $d \in \mathcal{D}$, and thus (6) provides valid bounds on $\theta(Q)$ under the stated assumptions.

(6) $\subseteq \Theta_0(P, \mathbf{Q})$

We now show that the bounds of (6) are the identified set for $\theta(Q)$, that is, the bounds of (6) are a subset of $\Theta_0(P, \mathbf{Q})$. Let q denote latent probabilities corresponding to a fixed $Q \in \mathbf{Q}_0(P, \mathbf{Q})$. There exists such a q by the assumption that $\mathbf{Q}_0(P, \mathbf{Q})$ is non-empty. We show that for each θ_0 in the right-hand side of (6), we can construct an alternative distribution of potential outcomes and treatments Q^* that is contained in $\mathbf{Q}_0(P, \mathbf{Q})$ and for which $\theta(Q^*) = (\mathbb{E}_{Q^*}[Y_j] : j \in \mathcal{D})$ is equal to θ_0. In particular, for each θ_0 in the right-hand side of (6) we will construct q^* corresponding to Q^* that

(a) satisfies $\sum_{r^o} q^*(r^o, r^t) = \sum_{r^o} q(r^o, r^t)$ and hence $Q^* \in \mathbf{Q}$ because the distribution of $(D_z : z \in \mathcal{Z})$ is unchanged and by assumption that $\mathbf{Q}$ satisfies Assumption 3.1,

(b) satisfies (S.2) and hence $P = Q^* T^{-1}$ due to Lemma A.1, and

(c) satisfies $\theta(Q^*) = \theta_0$.

Properties (a) and (b) allow us to conclude that $Q^* \in \mathbf{Q}_0(P, \mathbf{Q})$, that is, the constructed distribution is consistent with P and the model $\mathbf{Q}$. These properties will follow from our iterative construction of q^*, which preserves the marginal distribution of potential treatments but modifies the marginal distributions of potential outcomes for outcomes that are never observed for a given treatment response type, that is, correspond to a never-taken treatment for a given treatment response type. Because the marginal distribution of potential treatments is preserved, property (a) follows. Because only the marginal distributions of potential outcomes for never-taken treatments are modified, property (b) follows. Property (c) follows from being able to flexibly modify the marginal distributions of potential outcomes for never-taken treatments, so that any θ can be achieved.

Part 1: construct Q^*

We now construct an alternative q^* as follows. Fix some vector of weights $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{|\mathcal{D}|-1})' \in [0,1]^{|\mathcal{D}|}$ to be specified below. For each treatment response type r^t, let $q_0^*(r^o, r^t) = q(r^o, r^t)$ for all r^o. Let $K(r^t) = |\mathcal{N}(r^t)|$ be the number of treatments for which treatment response type r^t is a never-taker. Note that if $K(r^t) = 0$, then $\mathcal{N}(r^t) = \emptyset$, so $r^t_{z^*(d)} = d$ for all $d \in \mathcal{D}$ under r^t. For such an r^t, we set $q^*(r^o, r^t) = q_0^*(r^o, r^t) = q(r^o, r^t)$ for all $r^o \in \mathcal{Y}^{|\mathcal{D}|}$.

If $K(r^t) \geq 1$, enumerate the set of never-taken treatments $\mathcal{N}(r^t)$ as $\{j[1], ..., j[K(r^t)]\}$, and for $k = 1$ to $K(r^t)$, define q_k^* iteratively as follows:

$$\begin{aligned}
q_k^*((r^o_{-j[k]}, r^o_{j[k]} = y^L), r^t) &= (1 - \alpha_{j[k]}) \sum_{r^o_{j[k]} \in \mathcal{Y}} q_{k-1}^*((r^o_{-j[k]}, r^o_{j[k]}), r^t) \\
q_k^*((r^o_{-j[k]}, r^o_{j[k]} = y), r^t) &= 0 \qquad \text{for } y \notin \{y^L, y^U\} \\
q_k^*((r^o_{-j[k]}, r^o_{j[k]} = y^U), r^t) &= \alpha_{j[k]} \sum_{r^o_{j[k]} \in \mathcal{Y}} q_{k-1}^*((r^o_{-j[k]}, r^o_{j[k]}), r^t) \ ,
\end{aligned} \tag{S.3}$$

for all $r^o_{-j[k]}$, where we partition $r^o = (r^o_{-j[k]}, r^o_{j[k]})$. Intuitively, in step k, for each $r^o_{-j[k]}$ and r^t we reassign the probabilities of all outcome responses to never-taken treatment $j[k]$ to outcome responses y^L and y^U, splitting between y^L and y^U according to weight $\alpha_{j[k]}$.

With this construction, the marginal distribution of $Y_{j[k]}$ for treatment response type r^t is only modified in step k. This statement implies that for each fixed k, for step $\ell \leq k-1$, and for any outcome $y \in \mathcal{Y}$,

$$\sum_{r^o_{-j[k]}} q^*_\ell((r^o_{-j[k]}, r^o_{j[k]} = y), r^t) = \sum_{r^o_{-j[k]}} q^*_0((r^o_{-j[k]}, r^o_{j[k]} = y), r^t) = \sum_{r^o_{-j[k]}} q((r^o_{-j[k]}, r^o_{j[k]} = y), r^t) \ . \tag{S.4}$$

On the other hand, for each fixed k, because the marginal distribution of $Y_{j[k]}$ for treatment response type r^t is not further modified after step k, (S.3) and (S.4) imply

$$\begin{aligned}
\sum_{r^o_{-j[k]}} q^*_{K(r^t)}((r^o_{-j[k]}, r^o_{j[k]} = y^L), r^t) &= \sum_{r^o_{-j[k]}} q^*_k((r^o_{-j[k]}, r^o_{j[k]} = y^L), r^t) \\
&= (1-\alpha_{j[k]}) \sum_{r^o_{j[k]} \in \mathcal{Y}} \sum_{r^o_{-j[k]}} q((r^o_{-j[k]}, r^o_{j[k]}), r^t) \\
&= (1-\alpha_{j[k]}) \sum_{r^o} q(r^o, r^t) \ , \\
\sum_{r^o_{-j[k]}} q^*_{K(r^t)}((r^o_{-j[k]}, r^o_{j[k]} = y), r^t) &= 0 \qquad \text{for } y \notin \{y^L, y^U\} \ , \\
\sum_{r^o_{-j[k]}} q^*_{K(r^t)}((r^o_{-j[k]}, r^o_{j[k]} = y^U), r^t) &= \sum_{r^o_{-j[k]}} q^*_k((r^o_{-j[k]}, r^o_{j[k]} = y^U), r^t) \\
&= \alpha_{j[k]} \sum_{r^o_{j[k]} \in \mathcal{Y}} \sum_{r^o_{-j[k]}} q((r^o_{-j[k]}, r^o_{j[k]}), r^t) \\
&= \alpha_{j[k]} \sum_{r^o} q(r^o, r^t) \ .
\end{aligned} \tag{S.5}$$

These equations state that under $q^*_{K(r^t)}$, the constructed distribution in the final step $K(r^t)$, the probability that $r^o_{j[k]} = y$ for each r^t is zero if y is not y^L or y^U, is $\alpha_{j[k]}$ times the true probability of r^t under q if $y = y^U$, and is $1 - \alpha_{j[k]}$ times the true probability of r^t under q if $y = y^L$.

Finally, set

$$q^*(r^o, r^t) = q^*_{K(r^t)}(r^o, r^t) \qquad \forall\ r^o \ .$$

Part 2: verify property (a), $Q^* \in \mathbf{Q}$

With this construction, $q^*(r^o, r^t)$ is non-negative and from (S.5) we have

$$\sum_{r^o} q^*(r^o, r^t) = \sum_{r^o} q(r^o, r^t) \quad \text{for all} \quad r^t \ .$$

Because we assume $\mathbf{Q}$ satisfies Assumption 3.1 and $Q \in \mathbf{Q}$, this implies that $Q^* \in \mathbf{Q}$.

Part 3: verify property (b), $P = Q^* T^{-1}$

Furthermore, for each r^t and for all $r^o_c(r^t)$, the construction of q^* in (S.3) implies

$$\begin{aligned}
q^*(r^o_c(r^t), r^t) &= \sum_{r^o_n(r^t) \in \mathcal{Y}^{|\mathcal{N}(r^t)|}} q^*(r^o_n(r^t), r^o_c(r^t), r^t) \\
&= \sum_{r^o_{j[1]} \in \mathcal{Y}} \cdots \sum_{r^o_{j[K(r^t)]} \in \mathcal{Y}} q^*_{K(r^t)}((r^o_{j[1]}, \ldots, r^o_{j[K(r^t)]}), r^o_c(r^t), r^t) \\
&= \sum_{r^o_{j[1]} \in \mathcal{Y}} \cdots \sum_{r^o_{j[K(r^t)]} \in \{y^L, y^U\}} q^*_{K(r^t)}(r^o_{-j[K(r^t)]}, r^o_{j[K(r^t)]}, r^t) \\
&= \sum_{r^o_{j[1]} \in \mathcal{Y}} \cdots \sum_{r^o_{j[K(r^t)]} \in \mathcal{Y}} q^*_{K(r^t)-1}(r^o_{-j[K(r^t)]}, r^o_{j[K(r^t)]}, r^t) \\
&= \sum_{r^o_{j[1]} \in \mathcal{Y}} \cdots \sum_{r^o_{j[K(r^t)]} \in \mathcal{Y}} q^*_0((r^o_{j[1]}, \ldots, r^o_{j[K(r^t)]}), r^o_c(r^t), r^t) \\
&= \sum_{r^o_n(r^t) \in \mathcal{Y}^{|\mathcal{N}(r^t)|}} q(r^o_n(r^t), r^o_c(r^t), r^t) \\
&= q(r^o_c(r^t), r^t) \ .
\end{aligned}$$

Therefore for each r^t, $q^*(r^o_c(r^t), r^t) = q(r^o_c(r^t), r^t) \ \ \forall \ r^o_c(r^t)$, and hence by Lemma A.1, q^* satisfies (S.1) so that $P = Q^* T^{-1}$. Thus $Q^* \in \mathbf{Q}_0(P, \mathbf{Q})$.

Part 4: verify property (c), $\theta(Q^*) = \theta_0$

Note for each $d \in \mathcal{D}$,

$$\mathbb{E}_{Q^*}[Y_d \mathbb{1}\{D_{z^*(d)} = d\}] = \mathbb{E}_P[Y \mathbb{1}\{D = d\} \mid Z = z^*(d)] = \beta_{d|z^*(d)} \ .$$

Further note since Assumption 2.2 holds for Q, we have that for each $d \in \mathcal{D}$ and for each r^t if $r^t_{z^*(d)} = d$ then r^o_d is a component of $r^o_c(r^t)$ and $d \in \mathcal{N}(r^t)^c$, while if $r^t_{z^*(d)} \neq d$ then r^o_d

is a component of $r_n^o(r^t)$ and $d \in \mathcal{N}(r^t)$. Then we also have that for each $d \in \mathcal{D}$,

$$\begin{aligned}
\mathbb{E}_{Q^*}[Y_d \mathbb{1}\{D_{z^*(d)} \neq d\}] &= \sum_{y \in \mathcal{Y}} \sum_{r^t : r^t_{z^*(d)} \neq d} \sum_{r^o : r^o_d = y} y \; q^*(r^o, r^t) \\
&= \sum_{y \in \mathcal{Y}} \sum_{r^t : r^t_{z^*(d)} \neq d} \sum_{r^o_{-d}} y \; q^*((r^o_{-d}, r^o_d = y), r^t) \\
&= (\alpha_d y^U + (1-\alpha_d) y^L) \sum_{r^t : r^t_{z^*(d)} \neq d} \sum_{r^o} q(r^o, r^t) \\
&= (\alpha_d y^U + (1-\alpha_d) y^L) \; Q\{D_{z^*(d)} \neq d\} \\
&= (\alpha_d y^U + (1-\alpha_d) y^L)(1 - \sum_y p_{yd|z^*(d)}) \; ,
\end{aligned}$$

where the third equality is using that (S.5) holds for r^t such that $r^t_{z^*(d)} \neq d$, so that $d = j[k']$ for some k' in constructing $q^*(\cdot, r^t)$ for that r^t, and the last equality is using that Q satisfies (S.1). Thus, for each $d \in \mathcal{D}$,

$$\begin{aligned}
\mathbb{E}_{Q^*}[Y_d] &= \mathbb{E}_{Q^*}[Y_d \mathbb{1}\{D_{z^*(d)} = d\}] + \mathbb{E}_{Q^*}[Y_d \mathbb{1}\{D_{z^*(d)} \neq d\}] \\
&= \beta_{d|z^*(d)} + (\alpha_d y^U + (1-\alpha_d) y^L)(1 - \sum_y p_{yd|z^*(d)}) \; .
\end{aligned}$$

For any θ_0 contained in (6), we can thus choose $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{|\mathcal{D}|-1})' \in [0,1]^{|\mathcal{D}|}$ such that $\theta(Q^*) = \theta_0$. ■

A.4 Proof of Corollary 3.1

The results follows immediately from Theorem 3.2 because $\mathbb{E}[Y_j] - \mathbb{E}[Y_k]$ is a function of $\theta(Q)$. ■

A.5 Proof of Lemma 3.1

To see it, note for any $d \in \mathcal{D}$ and $z \in \mathcal{Z}$,

$$\mathbb{E}_Q[Y_d] = \mathbb{E}_Q[Y_d \mid Z = z] = \mathbb{E}_Q[Y_d \mathbb{1}\{D = d\} \mid Z = z] + \mathbb{E}_Q[Y_d \mathbb{1}\{D \neq d\} \mid Z = z]$$

$$
\begin{aligned}
&= \mathbb{E}_Q[Y\mathbb{1}\{D = d\} \mid Z = z] + \mathbb{E}_Q[Y_d\mathbb{1}\{D \neq d\} \mid Z = z] \\
&= \beta_{d|z} + \mathbb{E}_Q[Y_d\mathbb{1}\{D \neq d\} \mid Z = z] \\
&\leq \beta_{d|z} + y^U P\{D \neq d \mid Z = z\} \\
&= \beta_{d|z} + y^U(1 - \sum_{y\in\mathcal{Y}} p_{yd|z}) \ .
\end{aligned}
$$

Because the inequality holds for all $z \in \mathcal{Z}$, the upper end for each $d \in \mathcal{D}$ of (9) is a valid upper bound for $\mathbb{E}[Y_d]$. On the other hand, they can be simultaneously attained for all $d \in \mathcal{D}$ by setting $Y_d = y^U$ whenever $D \neq d$ and $Z = z$, without affecting the distribution of (Y, D, Z). A similar argument can be applied to the lower ends. In addition, any values in between can also be attained simultaneously for all $d \in \mathcal{D}$ by setting Y_d to be a convex combination of y^L and y^U whenever $D \neq d$ and $Z = z$ without affecting the distribution of (Y, D, Z), and therefore (9) is indeed the identified set for $\theta(Q)$ under mean independence.

■

A.6 Proof of Lemma 3.2

Suppose $Q \in \mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$. Consider the upper endpoints of (6) and (9). For any $d \in \mathcal{D}$, $z \in \mathcal{Z}$,

$$
\begin{aligned}
&\mathbb{E}_P[Y\mathbb{1}\{D = d\}] \mid Z = z^*(d)] + y^U\mathbb{E}_P[1 - \mathbb{1}\{D = d\} \mid Z = z^*(d)] \\
&\qquad - \mathbb{E}_P[Y\mathbb{1}\{D = d\} \mid Z = z] - y^U\mathbb{E}_P[1 - \mathbb{1}\{D = d\} \mid Z = z] \\
&= \mathbb{E}_Q[Y_d\mathbb{1}\{D_{z^*(d)} = d\}] + y^U\mathbb{E}_Q[1 - \mathbb{1}\{D_{z^*(d)} = d\}] \\
&\qquad - \mathbb{E}_Q[Y_d\mathbb{1}\{D_z = d\}] - y^U\mathbb{E}_Q[1 - \mathbb{1}\{D_z = d\}] \\
&= \mathbb{E}_Q[(Y_d - y^U)(\mathbb{1}\{D_{z^*(d)} = d\} - \mathbb{1}\{D_z = d\})] \\
&= \mathbb{E}_Q[(Y_d - y^U)(\mathbb{1}\{D_{z^*(d)} = d, D_z \neq d\} - \mathbb{1}\{D_{z^*(d)} \neq d, D_z = d\})] \\
&= \mathbb{E}_Q[(Y_d - y^U)\mathbb{1}\{D_{z^*(d)} = d, D_z \neq d\}]
\end{aligned}
$$

$$\leq 0 \ ,$$

where the first equality uses Assumption 2.1 and the fourth equality uses that $Q\{D_{z^*(d)} \neq d, D_z = d\} = 0$ for all Q satisfying Assumption 2.2. Since this inequality holds for all $z \in \mathcal{Z}$, we have that the upper endpoint of the interval in (6) is weakly smaller than the upper endpoint of the interval in (9). Conversely, the upper endpoint of (6) is contained in the set over which the upper endpoint of (9) is minimizing over, and thus the upper endpoint of (6) is weakly larger than the upper endpoint of (9). We conclude that the upper endpoints are the same. Parallel arguments show the equivalence of the lower endpoints. ■

A.7 Proof of Theorem 3.3

Because by assumption $\mathbf{Q} \subseteq \mathbf{Q}' \subseteq \mathbf{Q}^*_{MI}$, we have

$$\Theta_0(P, \mathbf{Q}) \subseteq \Theta_0(P, \mathbf{Q}') \subseteq \Theta_0(P, \mathbf{Q}^*_{MI}) \ .$$

By Lemma 3.2, if $\mathbf{Q} \subseteq \mathbf{Q}^*_{E,M}$, $\mathbf{Q}$ satisfies Assumption 3.1, and $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$, then we have

$$\Theta_0(P, \mathbf{Q}) = \Theta_0(P, \mathbf{Q}^*_{MI}) \ .$$

The result now follows by a sandwich argument. ■

A.8 Proof of Corollary 3.2

The result follows by taking $\mathbf{Q} = \mathbf{Q}^*_{E,M}$ and $\mathbf{Q}' = \mathbf{Q}^*_E$, noting that $\mathbf{Q}^*_E \subseteq \mathbf{Q}^*_{MI}$ because Assumption 2.1 implies Assumption 3.2. ■

A.9 Proof of Corollary 3.3

To begin, note by assumption $\mathbf{Q}_0(P, \mathbf{Q}^*_{WE,M}) \neq \emptyset$, so there exists $Q \in \mathbf{Q}^*_{WE,M}$ such that $P = QT^{-1}$. We then have $Q\{Z = z\} = P\{Z = z\}$ for all $z \in \mathcal{Z}$, and because Q

satisfies Assumption 3.3, we know (2) holds. Let Q_1 denote the marginal distribution of $((Y_d : d \in \mathcal{D}), (D_z : z \in \mathcal{Z}))$ under Q, and Q_Z denote the marginal distribution of Z under Q, and define $\tilde{Q} = Q_1 \times Q_Z$. Then, $\tilde{Q}$ satisfies Assumption 2.1 by construction, and it satisfies Assumption 2.2 because the marginal distribution of $(D_z : z \in \mathcal{Z})$ under $\tilde{Q}$ is the same as that under Q. In summary, $\tilde{Q} \in \mathbf{Q}^*_{E,M}$. Furthermore, $P = \tilde{Q}T^{-1}$ because (1) $\tilde{Q}\{Z = z\} = Q\{Z = z\} = P\{Z = z\}$ for all $z \in \mathcal{Z}$; and (2) $\tilde{Q}\{Y_d = y, D_z = d\} = Q\{Y_d = y, D_z = d\}$ for all $d \in \mathcal{D}$ and $z \in \mathcal{Z}$, and hence (2) is still satisfied. As a result, we know $\mathbf{Q}_0(P, \mathbf{Q}^*_{E,M}) \neq \emptyset$.

Next, take $\mathbf{Q} = \mathbf{Q}^*_{E,M}$ and $\mathbf{Q}' = \mathbf{Q}^*_{WE,M}$, and note that $\mathbf{Q}^*_{E,M} \subseteq \mathbf{Q}^*_{WE,M} \subseteq \mathbf{Q}^*_{MI}$ because Assumption 2.1 implies Assumption 3.3, which in turn implies Assumption 3.2. Because we know $\mathbf{Q}_0(P, \mathbf{Q}^*_{E,M}) \neq \emptyset$ from the previous paragraph, we then obtain from Theorem 3.3 that

$$\Theta_0(P, \mathbf{Q}^*_{E,M}) = \Theta_0(P, \mathbf{Q}^*_{WE,M}) = \Theta_0(P, \mathbf{Q}^*_{MI}) \ .$$

Similarly, taking $\mathbf{Q}' = \mathbf{Q}^*_{WE}$, we have

$$\Theta_0(P, \mathbf{Q}^*_{E,M}) = \Theta_0(P, \mathbf{Q}^*_{WE}) = \Theta_0(P, \mathbf{Q}^*_{MI}) \ .$$

The desired conclusion now follows. ■

B Details of Examples

B.1 Details of Example 4.5

We show that, except in the special case where the treatment and the instrument are both binary, the strict one-to-one targeting assumption of Lee and Salanié (2023) with one or more targeted treatments implies that (12) does not hold for some treatments. To see this, suppose that the strict one-to-one targeting assumption of Lee and Salanié (2023) holds with $|\mathcal{D}^\dagger| \geq 1$. For each $d \in \mathcal{D}^\dagger$, their assumptions include that there exists some $z^\dagger(d)$ and some $\overline{U}(d)$, $\underline{U}(d)$ with $\overline{U}(d) > \underline{U}(d)$ such that

$$g(z,d) = \begin{cases} \overline{U}(d) & \text{if } z = z^\dagger(d) \\ \underline{U}(d) & \text{if } z \neq z^\dagger(d) \ . \end{cases} \tag{S.6}$$

On the other hand, for each $d \in \mathcal{D} \setminus \mathcal{D}^\dagger$, they impose that $g(z,d) = \underline{U}(d)$ for all $z \in \mathcal{Z}$, and they impose that there is at least one such non-targeted treatment. For any $d \in \mathcal{D} \setminus \mathcal{D}^\dagger$, (12) requires that there exists $z^*(d) \in \mathcal{Z}$ such that

$$g(z,d') > g(z^*(d),d') \quad \text{for all } d' \neq d \text{ and } z \neq z^*(d) \ . \tag{S.7}$$

Suppose $|\mathcal{Z}| \geq 3$, and fix some targeted treatment $d' \in \mathcal{D}^\dagger$. Suppose $z^*(d) \neq z^\dagger(d')$. Then, for $z \in \mathcal{Z} \setminus \{z^*(d), z^\dagger(d')\}$, (S.7) requires $\underline{U}(d') > \underline{U}(d')$, a contradiction. Now suppose $z^*(d) = z^\dagger(d')$. Then, for $z \in \mathcal{Z} \setminus \{z^*(d)\}$, (S.7) requires $\underline{U}(d') > \overline{U}(d')$, a contradiction. Thus, $|\mathcal{Z}| \geq 3$ implies that (12) does not hold for non-targeted treatments.

Now suppose $|\mathcal{Z}| = 2$, which we label as $\mathcal{Z} = \{0,1\}$, and suppose $|\mathcal{D}| \geq 3$. Without loss of generality suppose $1 \in \mathcal{D}^\dagger$ and $z^\dagger(1) = 1$. If $z^*(d) = 1$, then (S.7) requires $\underline{U}(1) > \overline{U}(1)$, a contradiction. Now suppose $z^*(d) = 0$. Then (S.7) requires $g(1,d') > g(0,d')$ for all $d' \neq d$. Consider the following two cases:

- If $|\mathcal{D}^\dagger| = 1$, then $g(1,d') > g(0,d')$ holding for any $d' \in (\mathcal{D} \setminus \mathcal{D}^\dagger) \setminus \{d\}$ requires $\underline{U}(d') > \underline{U}(d')$, a contradiction.

- If $|\mathcal{D}^\dagger| > 1$, then there exists $d'' \in D^\dagger \setminus \{1\}$. By assumption $z^\dagger(d'') \neq z^\dagger(1)$ so that $z^\dagger(d'') = 0$. Then $g(1,d') > g(0,d')$ holding for $d' = d''$ requires $\underline{U}(d'') > \overline{U}(d'')$, again a contradiction.

Thus, $|\mathcal{Z}| = 2$ with $|\mathcal{D}| \geq 3$ implies that (12) does not hold for some treatments.

We have shown (12) does not hold for some treatments when either Z or D takes at least three values. Now suppose $|\mathcal{D}| = |\mathcal{Z}| = 2$. Let $D = 0$ denote the nontargeted treatment and $D = 1$ the targeted treatment, and let $z^\dagger(1) = 1$. Consider $z^*(0) = 0$ and $z^*(1) = 1$. Then evaluating (12) at either $d = 0$ or $d = 1$ results in $\overline{U}(1) > \underline{U}(1)$, and thus (12) holds when $|\mathcal{D}| = |\mathcal{Z}| = 2$. We conclude that the strict one-to-one targeting of Lee and Salanié (2023) implies that (12) does not hold for some $d \in \mathcal{D}$ except in the special case where $|\mathcal{D}| = |\mathcal{Z}| = 2$.

We now show that the strict one-to-one targeting of Lee and Salanié (2023) implies that Assumption 2.2 holds when $|\mathcal{Z}| > |\mathcal{D}^\dagger|$. Let $\mathcal{Z}^\dagger \subseteq \mathcal{Z}$ denote the set of instruments that target some treatment,

$$\mathcal{Z}^\dagger = \left\{z \in \mathcal{Z} : z = z^\dagger(d) \text{ for some } d \in \mathcal{D}^\dagger\right\} .$$

Their strict one-to-one targeting assumption combined with $|\mathcal{Z}| > |\mathcal{D}^\dagger|$ implies that there are values of the instrument that do not target any treatment; in other words, $\mathcal{Z}^\dagger \subsetneqq \mathcal{Z}$. Following Lee and Salanié (2023), we label the treatment that is known not to be targeted as treatment 0, so that $g(z, 0) = \underline{U}(0)$ for all $z \in \mathcal{Z}$, and impose their normalization that $\underline{U}(0) = 0$. Consider (4) for $d = 0$. Note that

$$Q\{D_{z^*(0)} \neq 0,\ D_{z'} = 0 \text{ for some } z' \neq z^*(0)\} = Q\left\{\bigcup_{d^* \neq 0, z' \neq z^*(0)} D_{z^*(0)} = d^*,\ D_{z'} = 0\right\} .$$

We wish to investigate whether there exists some $z^*(0) \in \mathcal{Z}$ such that the above probability is zero. Consider $z^*(0)$ equal to any value in $\mathcal{Z} \setminus \mathcal{Z}^\dagger$, i.e., any value of the instrument that does not target any treatment. For any fixed $d^* \neq 0, z' \neq z^*(0)$, consider the event $\{D_{z^*(0)} = d^*,\ D_{z'} = 0\}$. Since $z^*(0)$ does not target any treatment and thus does not target d^*, $D_{z^*(0)} = d^*$ implies

$$U_0 - U_{d^*} \leq \underline{U}(d^*) . \tag{S.8}$$

If z' targets d^*, then $D_{z'} = 0$ implies

$$U_0 - U_{d^*} \geq \overline{U}(d^*) \ . \tag{S.9}$$

If z' does not target d^*, then $D_{z'} = 0$ implies

$$U_0 - U_{d^*} \geq \underline{U}(d^*) \ . \tag{S.10}$$

Thus, the event $\{D_{z^*(0)} = d^*,\ D_{z'} = 0\}$ either requires (S.8) and (S.9) to jointly hold, which is a contradiction since $\overline{U}(d^*) > \underline{U}(d^*)$, or requires (S.8) and (S.10) to jointly hold, which is a zero probability event given our assumption that the distribution of $(U_d : d \in \mathcal{D})$ is absolutely continuous w.r.t. Lebesgue measure. Thus $Q\{D_{z^*(0)} \neq 0,\ D_{z'} = 0 \text{ for some } z' \neq z^*(0)\}$ is a probability of a finite union of zero probability events, and thus, by Boole's inequality, equals zero so that (4) holds for $d = 0$. A parallel argument shows that (4) holds for any non-targeted treatment, and related argument shows that (4) holds for any targeted treatment. Thus, under the strict one-to-one targeting of Lee and Salanié (2023), when there are more values of the instrument than targeted treatments, Assumption 2.2 holds even though (12) is violated for some $d \in \mathcal{D}$.

B.2 Details of Example 5.1

Let $\mathbf{Q}$ denote all distributions Q for which Assumption 2.1 holds and such that $Q\{D_0 = D_1\} = 0$. Then for $Q \in \mathbf{Q}_0(P, \mathbf{Q})$,

$$p_{y1|1} = Q\{Y_1 = y, D_1 = 1, D_0 = 0\}$$

$$p_{y0|0} = Q\{Y_0 = y, D_1 = 1, D_0 = 0\}$$

$$p_{y0|1} = Q\{Y_0 = y, D_1 = 0, D_0 = 1\}$$

$$p_{y1|0} = Q\{Y_1 = y, D_1 = 0, D_0 = 1\}$$

and

$$
\begin{aligned}
Q\{Y_0 = 1\} &= Q\{Y_0 = 1, D_1 = 1, D_0 = 0\} + Q\{Y_0 = 1, D_1 = 0, D_0 = 1\} \\
&= p_{10|0} + p_{10|1} \\
Q\{Y_1 = 1\} &= Q\{Y_1 = 1, D_1 = 1, D_0 = 0\} + Q\{Y_1 = 1, D_1 = 0, D_0 = 1\} \\
&= p_{11|1} + p_{11|0} \ .
\end{aligned}
$$

Therefore, if $\mathbf{Q}$ is consistent with P, then $\theta(Q)$ is identified as

$$
\Theta_0(P, \mathbf{Q}) = \left\{ \begin{pmatrix} p_{10|0} + p_{10|1} \\ p_{11|0} + p_{11|1} \end{pmatrix} \right\} \ . \tag{S.11}
$$

In contrast, the identified set that follows from imposing Assumption 2.1 alone, $\Theta_0(P, \mathbf{Q}_E^*)$, is shown by Balke and Pearl (1997) to be

$$
\max \left\{ \begin{matrix} p_{10|1} \\ p_{10|0} \\ p_{10|0} + p_{11|0} - p_{00|1} - p_{11|1} \\ p_{01|0} + p_{10|0} - p_{00|1} - p_{01|1} \end{matrix} \right\} \leq Q\{Y_0 = 1\} \leq \min \left\{ \begin{matrix} 1 - p_{00|1} \\ 1 - p_{00|0} \\ p_{01|0} + p_{10|0} + p_{10|1} + p_{11|1} \\ p_{10|0} + p_{11|0} + p_{01|1} + p_{10|1} \end{matrix} \right\} \tag{S.12}
$$

and

$$
\max \left\{ \begin{matrix} p_{11|0} \\ p_{11|1} \\ -p_{00|0} - p_{01|0} + p_{00|1} + p_{11|1} \\ -p_{01|0} - p_{10|0} + p_{10|1} + p_{11|1} \end{matrix} \right\} \leq Q\{Y_1 = 1\} \leq \min \left\{ \begin{matrix} 1 - p_{01|1} \\ 1 - p_{01|0} \\ p_{00|0} + p_{11|0} + p_{10|1} + p_{11|1} \\ p_{10|0} + p_{11|0} + p_{00|1} + p_{11|1} \end{matrix} \right\} \ . \tag{S.13}
$$

It follows from $\mathbf{Q} \subseteq \mathbf{Q}_E^*$ that $\Theta_0(P, \mathbf{Q}) \subseteq \Theta_0(P, \mathbf{Q}_E^*)$, and thus (S.11) is contained in (S.12)–(S.13).

Next, we show that there exists a P for which $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$, $\Theta_0(P, \mathbf{Q})$ is not given by (6) and $\Theta_0(P, \mathbf{Q}) \subsetneqq \Theta_0(P, \mathbf{Q}^*_E)$. We do so by providing a numerical example. Consider the P specified in Table 1 and the Q specified in Table 2, where we write $q(y_0 y_1, d_0 d_1) = Q\{Y_d = y_d, D_z = d_z,\ (d, z) \in \mathcal{D} \times \mathcal{Z}\}$ and omit any $q(\cdot) = 0$. One can check that $Q \in \mathbf{Q}$ and rationalizes P, so that $Q \in \mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$. In this example, $Q\{Y_0 = 1\} = 0.4274$, and thus the identified set for $Q\{Y_0 = 1\}$ relative to $\mathbf{Q}$ is the singleton $\{0.4274\}$. In contrast, evaluating (S.12) at P gives the identified set for $Q\{Y_0 = 1\}$ relative to $\mathbf{Q}^*_E$ as $[0.3336, 0.5212]$. We thus conclude that $\Theta_0(P, \mathbf{Q}) \subsetneqq \Theta_0(P, \mathbf{Q}^*_E)$ for some P that can be rationalized by $Q \in \mathbf{Q}$. Now consider evaluating the bounds of (6) at the same P. Doing so results in bounds on $Q\{Y_0 = 1\}$ given by $[0.1618, 0.5445]$ if setting $z^*(0) = 0$ and given by $[0.2656, 0.8829]$ if setting $z^*(0) = 1$. Therefore, no matter $z^*(0) = 0$ or 1, the bounds (6) is not the identified set for $Q\{Y_0 = 1\}$ relative to either $\mathbf{Q}^*_E$ or $\mathbf{Q}$.

$p_{00\|0}$	$p_{10\|0}$	$p_{01\|0}$	$p_{11\|0}$
0.4555	0.1618	0.3077	0.0750
$p_{00\|1}$	$p_{10\|1}$	$p_{01\|1}$	$p_{11\|1}$
0.1171	0.2656	0.0188	0.5985

Table 1: Distribution P in Appendix B.2.

$q(00, 01)$	$q(00, 10)$	$q(01, 01)$	$q(01, 10)$
0.0039	0.0428	0.4516	0.0743
$q(10, 01)$	$q(10, 10)$	$q(11, 01)$	$q(11, 10)$
0.0149	0.2649	0.1469	0.0007

Table 2: Distribution Q.

B.3 Details of Example 5.2

Suppose $\mathcal{Y} = \{0, 1\}$, $\mathcal{D} = \{0, 1, 2\}$, and $\mathcal{Z} = \{0, 1, 2\}$. Then, the linear program approach in Balke and Pearl (1993, 1997) leads to the following identified set for $\mathbb{E}_Q[Y_1] = Q\{Y_1 = 1\}$ relative to $\mathbf{Q}$ being defined as in Example 5.2:

$$\left[\max\left\{\begin{array}{c} p_{11|0} \\ p_{11|1} \\ p_{11|2} \\ p_{11|0} - p_{11|1} + p_{11|2} \end{array}\right\}, \quad \min\left\{\begin{array}{c} 1 - p_{01|2} \\ 1 - p_{01|1} \\ 1 - p_{01|0} \\ 1 - p_{01|0} + p_{01|1} - p_{01|2} \end{array}\right\}\right] . \tag{S.14}$$

We will show that for some P such that $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$, we have $\Theta_0(P, \mathbf{Q})$ strictly smaller than (6) and $\Theta_0(P, \mathbf{Q}_1^*)$. For this purpose we are only concerned with the validity of (S.14) instead of its sharpness. For the lower bounds, first note for $z \in \mathcal{Z}$,

$$Q\{Y_1 = 1\} = Q\{Y_1 = 1 \mid Z = z\} \geq Q\{Y_1 = 1, D = 1 \mid Z = z\} = Q\{Y = 1, D = 1 \mid Z = z\} ,$$

and therefore the first three rows follow. To show the last row, note it's equivalent to

$$Q\{Y_1 = 1, D_1 = 1\} + Q\{Y_1 = 1, D_0 = 0\} + Q\{Y_1 = 1, D_0 = 2\} \geq Q\{Y_1 = 1, D_2 = 1\} .$$

It therefore suffices to show that

$$\{D_2 = 1\} \implies \{D_1 = 1\} \cup \{D_0 = 0\} \cup \{D_0 = 2\} . \tag{S.15}$$

Suppose $D_2 = 1$ but $D_0 \neq 0$ and $D_0 \neq 2$. Then $D_0 = 1$. But $D_0 \leq D_1 \leq D_2$, so $D_1 = 1$. (S.15) now follows. The lower bounds in (S.14) have all been shown to hold, and the upper bounds can be proved similarly.

Next, we show that there exists a P for which $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$, $\Theta_0(P, \mathbf{Q})$ is not given by (6) and $\Theta_0(P, \mathbf{Q}) \subsetneq \Theta_0(P, \mathbf{Q}_E^*)$. We do so by providing a numerical example. Consider the P specified in Table 3 and the four Q distributions specified in Table 4, 5, 6 and 7, which we denote as $Q_{\text{ex,min}}$, $Q_{\text{ex,max}}$, $Q_{\text{ex,om,min}}$ and $Q_{\text{ex,om,max}}$ respectively, where we write $q(y_0y_1y_2, d_0d_1d_2) = Q\{Y_d = y_d, D_z = d_z,\ (d,z) \in \mathcal{D} \times \mathcal{Z}\}$ and omit any $q(\cdot) = 0$. One can check that all the four Qs are in $\mathbf{Q}_0(P, \mathbf{Q}_E^*)$, i.e., they all rationalize P and satisfy Assumption 2.1. Moreover, $Q_{\text{ex,om,min}} \in \mathbf{Q}_0(P, \mathbf{Q})$ and $Q_{\text{ex,om,max}} \in \mathbf{Q}_0(P, \mathbf{Q})$ so that $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$. Evaluating (S.14) at P gives $[0.2117, 0.8205] =: I_{\text{ex,om}}$. In contrast, if one evaluates (6) by setting $z^*(1) = 0, 1, 2$ at the same P, the resulting bounds for $\mathbb{E}_Q[Y_1]$ are $[0.1664, 0.9255] =: I_{(6),0}$, $[0.0712, 0.9311] =: I_{(6),1}$ and $[0.1165, 0.8261] =: I_{(6),2}$ respectively. In all cases, we see $I_{\text{ex,om}} \subsetneq I_{(6),z^*(1)}$ for all $z^*(1) \in \{0, 1, 2\}$ so $\Theta_0(P, \mathbf{Q})$ is not given by (6). Furthermore, $Q_{\text{ex,min}} \notin \mathbf{Q}_0(P, \mathbf{Q})$ and $Q_{\text{ex,max}} \notin \mathbf{Q}_0(P, \mathbf{Q})$ because, for example, $q_{\text{ex,min}}(000, 021) > 0$ and $q_{\text{ex,max}}(000, 210) > 0$. At the same time, $\mathbb{E}_{Q_{\text{ex,min}}}[Y_1] = 0.1664 \notin I_{\text{ex,om}}$ and $\mathbb{E}_{Q_{\text{ex,max}}}[Y_1] = 0.8261 \notin I_{\text{ex,om}}$. Therefore, $\Theta_0(P, \mathbf{Q}) \subsetneq \Theta_0(P, \mathbf{Q}_E^*)$.

$p_{00\|0}$	$p_{10\|0}$	$p_{01\|0}$	$p_{11\|0}$	$p_{02\|0}$	$p_{12\|0}$
0.3808	0.2427	0.0745	0.1664	0.0345	0.1011
$p_{00\|1}$	$p_{10\|1}$	$p_{01\|1}$	$p_{11\|1}$	$p_{02\|1}$	$p_{12\|1}$
0.2830	0.1947	0.0689	0.0712	0.2014	0.1808
$p_{00\|2}$	$p_{10\|2}$	$p_{01\|2}$	$p_{11\|2}$	$p_{02\|2}$	$p_{12\|2}$
0.0802	0.0976	0.1739	0.1165	0.2444	0.2874

Table 3: Distribution P in Appendix B.3.

$q(000, 002)$	$q(000, 011)$	$q(000, 021)$	$q(001, 002)$	$q(001, 020)$	$q(001, 021)$	$q(001, 121)$	$q(001, 211)$
0.0139	0.0304	0.0054	0.2238	0.0802	0.0271	0.0735	0.0375
$q(010, 101)$	$q(010, 111)$	$q(010, 122)$	$q(100, 000)$	$q(100, 022)$	$q(100, 110)$	$q(100, 202)$	$q(101, 202)$
0.0453	0.0712	0.0499	0.0966	0.1461	0.0010	0.0345	0.0636

Table 4: Distribution $Q_{\text{ex,min}}$.

$q(001,021)$	$q(001,121)$	$q(001,211)$	$q(010,000)$	$q(010,001)$	$q(010,002)$	$q(010,010)$	$q(010,122)$
0.0305	0.0745	0.0689	0.0220	0.0064	0.0085	0.0260	0.1664
$q(010,202)$	$q(011,002)$	$q(011,022)$	$q(011,210)$	$q(110,000)$	$q(110,001)$	$q(110,011)$	$q(110,022)$
0.0345	0.2116	0.0758	0.0322	0.0976	0.0971	0.0130	0.0350

Table 5: Distribution $Q_{\text{ex,max}}$.

$q(000,000)$	$q(000,001)$	$q(000,002)$	$q(000,022)$	$q(000,111)$	$q(000,122)$	$q(000,222)$	$q(001,002)$
0.0802	0.0079	0.0430	0.0181	0.0209	0.0536	0.0345	0.1066
$q(001,022)$	$q(001,222)$	$q(010,001)$	$q(010,111)$	$q(010,122)$	$q(100,000)$	$q(100,001)$	$q(100,011)$
0.0797	0.1011	0.0453	0.0712	0.0952	0.0976	0.0971	0.0480

Table 6: Distribution $Q_{\text{ex,om,min}}$.

$q(000,001)$	$q(000,111)$	$q(000,122)$	$q(010,000)$	$q(010,001)$	$q(010,012)$	$q(010,111)$	$q(010,122)$
0.0079	0.0689	0.0056	0.0802	0.1114	0.0430	0.0051	0.1613
$q(010,222)$	$q(011,002)$	$q(011,022)$	$q(011,222)$	$q(100,001)$	$q(110,000)$	$q(111,012)$	$q(111,022)$
0.0345	0.0835	0.0548	0.1011	0.0971	0.0976	0.0231	0.0249

Table 7: Distribution $Q_{\text{ex,om,max}}$.

B.4 Details of Example 5.3

Consider the ARUM defined by (11) with strict one-to-one targeting and $|\mathcal{D}| = 3$, $|\mathcal{Z}| = |\mathcal{D}^\dagger| = 2$. In this case, there are two targeted treatments and one non-targeted treatment. Following Lee and Salanié (2023), label that non-targeted treatment as treatment 0 and impose the normalization that $g(z,0) = 0$ for all $z \in \mathcal{Z}$. Label $Z = 0$ as the instrument value that targets treatment 1 and label $Z = 1$ as the instrument value that targets treatment 2, so that (S.6) holds for $d = 1,2$ for some $\overline{U}(d), \underline{U}(d)$ with $\overline{U}(d) > \underline{U}(d)$ and with $z^\dagger(1) = 0$, $z^\dagger(2) = 1$. Let $\mathbf{Q}$ denote the set of all distributions for which $(D_z : z \in \mathcal{Z})$ is determined by (11) with these restrictions and additionally imposing that the support of (U_0, U_1, U_2) is $\Re^3$. Recall $(U_0, U_1, U_2) \perp\!\!\!\perp Z$ by assumption.

Let $U_{10} = U_1 - U_0$ and $U_{20} = U_2 - U_0$. In this model, the treatment value is completely determined by the vector of realizations (U_{10}, U_{20}). For instance, $D_z = 2$ if and only if

$$U_{20} \geq -g(z, 2)$$

$$U_{20} - U_{10} \geq g(z, 1) - g(z, 2) \ ,$$

and a similar characterization holds for $D_z = 1$. See Figure 1, which is taken from Figure 1 in Lee and Salanié (2023).

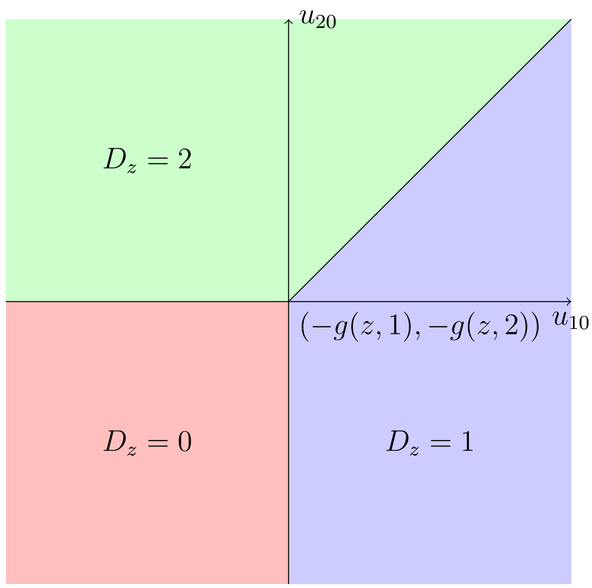

Figure 1: Treatment under each value of (u_{10}, u_{20}) for a given z.

We first show that (4) holds for the targeted treatments by verifying that (12) holds for the targeted treatments. Consider (12) for $d = 1$. It holds with $z^*(1) = z^\dagger(1) = 0$ because

$$\overline{U}(1) > \underline{U}(1) \ ,$$

$$\overline{U}(1) - \underline{U}(2) > \underline{U}(1) - \overline{U}(2) \ ,$$

which in turn holds because $\overline{U}(d) > \underline{U}(d)$ for $d \in \{1, 2\}$. Thus (12) holds for $d = 1$, which, as shown in Example 4.4, implies that (4) holds for $d = 1$. By a parallel argument, (4) holds for $d = 2$.

We now show that there does not exist a value of $z^*(0)$ such that (4) holds for the non-targeted treatment, treatment 0. Suppose $z^*(0) = 0$. Then

$$
\begin{aligned}
Q\{D_0 \neq 0,\ D_1 = 0\} \geq Q\{D_0 = 1,\ D_1 = 0\}& \\
&= Q\{-\underline{U}(1) \geq U_{10} \geq -\overline{U}(1), U_{20} \leq -\overline{U}(2), U_{10} - U_{20} \geq \underline{U}(2) - \overline{U}(1)\} \\
&> 0\ ,
\end{aligned}
$$

where the last line is using that the support of the distribution of $(U_{10}, U_{20}) = \Re^2$ by assumption and that strict targeting of treatment 1 requires $-\underline{U}(1) > -\overline{U}(1)$. Thus (4) cannot hold for $d = 0$ with $z^*(0) = 0$. A parallel argument shows that (4) cannot hold for $d = 0$ with $z^*(0) = 1$.

We conclude that, when $|\mathcal{D}| = 3$ and $|\mathcal{D}^\dagger| = |\mathcal{Z}| = 2$, one-to-one strict targeting with the regularity condition that the support of (U_0, U_1, U_2) is $\Re^3$ implies that (4) holds for the targeted treatments but not for the non-targeted treatments, and thus Assumption 2.2 cannot hold. This argument can be adapted for any ARUM with $|\mathcal{D}| \geq 3$ and $|\mathcal{Z}| = |\mathcal{D}^\dagger|$ to show that, while (4) holds for the targeted treatments, there does not exist a value of $z^*(d)$ such that (4) holds for any non-targeted treatment d, and thus that Assumption 2.2 cannot hold.

Next, consider the identified sets for the average potential outcomes. Since (4) is satisfied for the targeted treatments, a straightforward modification of the arguments underlying Theorem 3.2 show that the identified sets for $\mathbb{E}_Q[Y_d]$ for $d \in \{1, 2\}$ is given by (6) for any P such that $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$.

We now derive the identified set for the average potential outcome of the non-targeted

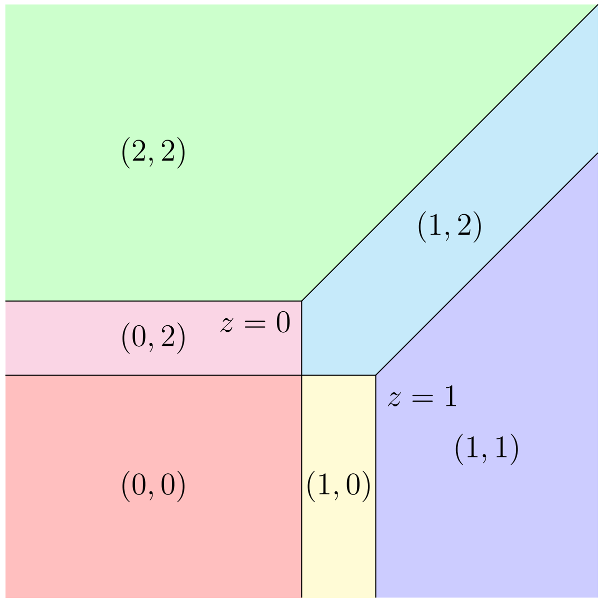

Figure 2: Values of (D_0, D_1) for each value of (u_{10}, u_{20}).

treatment. First, note that $-g(0,1) = -\overline{U}(1) < -\underline{U}(1) = -g(1,1)$ and $-g(0,2) = -\underline{U}(2) > -\overline{U}(2) = -g(1,2)$. Therefore, it can be verified from Figure 2 that for all $Q \in \mathbf{Q}$,

$$Q\{(D_0, D_1) \in \{(0,0),(1,0),(1,1),(0,2),(1,2),(2,2)\}\} = 1 \ . \tag{S.16}$$

Let $\mathbf{Q}'$ denote the set of all distributions that satisfies (S.16). Note that all $Q \in \mathbf{Q}$ satisfies (S.16), so $\mathbf{Q} \subseteq \mathbf{Q}'$. On the other hand, by assigning appropriate probabilities to each set in the partition in Figure 2, we immediately see that each $Q \in \mathbf{Q}'$ can be rationalized by a $Q \in \mathbf{Q}$. Therefore, $\mathbf{Q} = \mathbf{Q}'$. Using linear programming as in Balke and Pearl (1993, 1997), we obtain the following identified set for $\mathbb{E}_Q[Y_0] = Q\{Y_0 = 1\}$ relative to $\mathbf{Q}$:

$$\left[\max \left\{ \begin{matrix} p_{10|0} \\ \\ p_{10|1} \end{matrix} \right\}, \quad \min \left\{ \begin{matrix} 1 - p_{00|1} \\ \\ 1 - p_{00|0} \end{matrix} \right\} \right] \ . \tag{S.17}$$

The identified set in (S.17) equals (9) for $d = 0$ with Y and Z binary. Thus, the identified set for $\mathbb{E}_Q[Y_0]$ relative to $\mathbf{Q}$ corresponds to the identified set relative to $\mathbf{Q}_3^*$, the set of distributions that satisfy mean independence, 3.2. By the same sandwich argument used to prove Theorem 3.3, the identified set for $\mathbb{E}_Q[Y_0]$ relative to $\mathbf{Q}$ corresponds to the identified set relative to $\mathbf{Q}_1^*$, and thus imposing this ARUM has no identifying power for $\mathbb{E}_Q[Y_0]$ beyond instrument exogeneity.

Finally, we show that there exists a P for which $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$ and (S.17) is strictly smaller than (6), so that $\Theta_0(P, \mathbf{Q})$ is not given by (6). We do so by providing a numerical example. Consider the P specified in Table 8 and the $Q_{\text{arum,min}}$ and $Q_{\text{arum,max}}$ specified in Tables 9 and 10 respectively, where we write $q(y_0 y_1 y_2, d_0 d_1) = Q\{Y_d = y_d, D_z = d_z, \ (d, z) \in \mathcal{D} \times \mathcal{Z}\}$ and omit any $q(\cdot) = 0$. One can check that both $Q_{\text{arum,min}}$ and $Q_{\text{arum,max}}$ rationalize P and satisfy Assumption 2.1. One can further check that both $Q_{\text{arum,min}}$ and $Q_{\text{arum,max}}$ satisfy the restriction in (S.16), so that $\mathbf{Q}_0(P, \mathbf{Q}) \neq \emptyset$. Evaluating (S.17) at P gives the identified set for $\mathbb{E}_Q[Y_0]$ relative to $\mathbf{Q}$ as $[0.2518, 0.8167]$. One can further check that the two endpoints are attained by $\mathbb{E}_{Q_{\text{arum,min}}}[Y_0] = 0.2518$ and $\mathbb{E}_{Q_{\text{arum,max}}}[Y_0] = 0.8167$. On the other hand, if one evaluates (6) by setting $z^*(0) = 0, 1$ at the same P, the resulting bounds for $\mathbb{E}_Q[Y_0]$ equal $[p_{10|0}, 1 - p_{00|0}] = [0.2518, 0.8937]$ and $[p_{10|1}, 1 - p_{00|1}] = [0.2372, 0.8167]$ respectively. In both cases, (S.17) is strictly contained in (6). Hence, $\Theta_0(P, \mathbf{Q})$ is not given by (6).

$p_{00\|0}$	$p_{10\|0}$	$p_{01\|0}$	$p_{11\|0}$	$p_{02\|0}$	$p_{12\|0}$
0.1063	0.2518	0.2946	0.3183	0.0020	0.0270
$p_{00\|1}$	$p_{10\|1}$	$p_{01\|1}$	$p_{11\|1}$	$p_{02\|1}$	$p_{12\|1}$
0.1833	0.2372	0.0140	0.1399	0.1701	0.2555

Table 8: Distribution P in Appendix B.4.

$q(000, 10)$	$q(000, 11)$	$q(000, 12)$	$q(000, 22)$	$q(001, 02)$	$q(001, 12)$
0.0049	0.0140	0.1535	0.0020	0.1063	0.1222
$q(001, 22)$	$q(010, 10)$	$q(010, 11)$	$q(100, 00)$	$q(100, 02)$	
0.0270	0.1784	0.1399	0.2372	0.0146	

Table 9: Distribution $Q_{\text{arum,min}}$.

$q(000, 00)$	$q(010, 10)$	$q(100, 00)$	$q(100, 10)$	$q(100, 11)$	$q(100, 02)$
0.1063	0.0770	0.0837	0.0521	0.0140	0.1681
$q(100, 22)$	$q(101, 12)$	$q(101, 22)$	$q(110, 10)$	$q(110, 11)$	
0.0020	0.2285	0.0270	0.1014	0.1399	

Table 10: Distribution $Q_{\text{arum,max}}$.

C Additional Examples of Models That Satisfy Assumption 2.2

In Section 4, we considered examples of restrictions on potential treatments previously considered in the literature that satisfy generalized monotonicity. We now consider three additional such examples.

Example C.1. Consider the ARUM of Example 4.4 when $|\mathcal{D}| = 2$, and let $\mathbf{Q}$ denote the set of distributions defined in that example. Then, Assumption 2.2 holds for all $Q \in \mathbf{Q}$. To see this, consider $Q \in \mathbf{Q}$. Label $\mathcal{D} = \{0, 1\}$, and let $g_{10}(z) = g(z, 1) - g(z, 0)$ and $U_{10} = U_1 - U_0$. The assumptions of Example 4.4 on (U_1, U_0) imply that the distribution of U_{10} is absolutely continuous with respect to Lebesgue measure and that $U_{10} \perp\!\!\!\perp Z$. Ignoring ties that occur with probability zero, (11) can be rewritten as

$$D_z = \mathbb{1}\{g_{10}(z) + U_{10} \geq 0\} \ . \tag{S.18}$$

Let $\overline{\mathcal{Z}} = \operatorname{argmax}_{z \in \mathcal{Z}}\{g_{10}(z)\}$, and let $\underline{\mathcal{Z}} = \operatorname{argmin}_{z \in \mathcal{Z}}\{g_{10}(z)\}$. Then, Q satisfies Assumption 2.2 with $\mathcal{Z}^*(1) = \overline{\mathcal{Z}}$ and $\mathcal{Z}^*(0) = \underline{\mathcal{Z}}$. To contrast with Example 4.4, note that (12) holds if and only if $\overline{\mathcal{Z}}$ and $\underline{\mathcal{Z}}$ are both singletons. ■

Example C.2. Kline and Walters (2016) considers an RCT with a "close substitute" to study the effects of preschooling on educational outcomes. In their setting, $D \in \mathcal{D} = \{0, 1, 2\}$, where $D = 0$ denotes home care (no preschool), $D = 2$ denotes a preschool program called Head Start, and $D = 1$ denotes preschools other than Head Start, namely the close substitute. Let $Z \in \mathcal{Z} = \{0, 1\}$ denote an indicator variable for an offer to attend Head Start. Assumption 2.1 holds because Z is randomly assigned. Kline and Walters (2016) impose the restriction that

$$Q\{D_1 = 2 \mid D_0 \neq D_1\} = 1 \ . \tag{S.19}$$

The condition in (S.19) states that if the choice of a family changes upon receiving a Head Start offer, then they must choose Head Start when receiving the offer. In other words, it cannot be the case that upon receiving a Head Start offer, a family switches from no preschool to preschools other than Head Start, or the other way around. Assumption 2.2 then holds with $z^*(0) = z^*(1) = 0$ and $z^*(2) = 1$. To see this, note that (S.19) implies $Q\{D_0 \neq D_1, D_1 \neq 2\} = 0$ and thus

$$Q\{D_0 \neq 0, D_1 = 0\} = Q\{D_0 \neq D_1, D_1 = 0\} = 0 \ ,$$

$$Q\{D_0 \neq 1, D_1 = 1\} = Q\{D_0 \neq D_1, D_1 = 1\} = 0 \ ,$$

$$Q\{D_1 \neq 2, D_0 = 2\} \leq Q\{D_0 \neq D_1, D_1 \neq 2\} = 0 \ .$$

Note in this example Assumption 2.2 still holds although $|\mathcal{Z}| < |\mathcal{D}|$. See Bai et al. (2025) for results on the sharp testable implications of the assumptions for this example and Example C.3. ■

Example C.3. Kirkeboen et al. (2016) study the effects of fields of study on earnings. In their setting, $\mathcal{D} = \{0, 1, 2\}$ represent three fields of study, ordered by their (soft) admission cutoffs from the lowest to the highest. The instrument is $Z \in \{0, 1, 2\}$, with $Z = 1$ when the student crosses the (soft) admission cutoff for field 1, $Z = 2$ when the student crosses the (soft) admission cutoff for field 2, and $Z = 0$ otherwise. The authors assume that Z is exogenous in the sense that Q satisfies Assumption 2.1 and impose the following monotonicity conditions:

$$Q\{D_1 = 1 \mid D_0 = 1\} = 1 \ , \tag{S.20}$$

$$Q\{D_2 = 2 \mid D_0 = 2\} = 1 \ . \tag{S.21}$$

The conditions in (S.20)–(S.21) require that crossing the cutoff for field 1 or 2 weakly encourages them towards that field. They further impose the following "irrelevance" conditions:

$$Q\{\mathbb{1}\{D_1 = 2\} = \mathbb{1}\{D_0 = 2\} \mid D_0 \neq 1, D_1 \neq 1\} = 1 \ , \tag{S.22}$$

$$Q\{\mathbb{1}\{D_2 = 1\} = \mathbb{1}\{D_0 = 1\} \mid D_0 \neq 2, D_2 \neq 2\} = 1 \ . \tag{S.23}$$

The condition in (S.22) states that if crossing the cutoff for field 1 does not cause the student to switch to field 1, then it does not cause them to switch to or away from field 2. A similar interpretation applies to (S.23). Lee and Salanié (2023) show the set of all distributions that satisfy (S.20)–(S.23) are equivalent to a strict one-to-one targeting model with $|\mathcal{Z}| = 3$ and $|\mathcal{D}^\dagger| = 2$; it therefore follows from Remark 4.5 that any Q that satisfies (S.20)–(S.23) also satisfies Assumption 2.2. Here, we establish directly that (S.20)–(S.23) imply Assumption 2.2 with $z^*(0) = 0$, $z^*(1) = 1$, and $z^*(2) = 2$. To show $z^*(0) = 0$, we prove by contradiction that

$$Q\{D_0 \neq 0, D_1 = 0\} = 0 \ .$$

Suppose with positive probability that $D_0 \neq 0$ but $D_1 = 0$. On this event, (S.20) implies $D_0 \neq 1$, so $D_0 = 2$. But $D_1 = 0$, which contradicts (S.22). Similarly,

$$Q\{D_0 \neq 0, D_2 = 0\} = 0 \ ,$$

and therefore $z^*(0) = 0$. To show $z^*(1) = 1$, first note (S.20) implies

$$Q\{D_1 \neq 1, D_0 = 1\} = 0 \ .$$

It therefore remains to argue by contradiction that

$$Q\{D_1 \neq 1, D_2 = 1\} = 0 \ . \tag{S.24}$$

Suppose with positive probability that $D_1 \neq 1$ but $D_2 = 1$. On this event, (S.20) implies $D_0 \neq 1$. If $D_0 = 2$, then (S.21) implies $D_2 = 2$, a contradiction to $D_2 = 1$; if instead $D_0 = 0$, then because we assume $D_2 = 1$, (S.23) implies $D_2 \neq 1$, another contradiction. Therefore, (S.24) holds, and $z^*(1) = 1$. $z^*(2) = 2$ can be established following similar arguments. ■

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